

Active Particles Forced by an Asymmetric Dichotomous Angle Drive

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We analyze the dynamics of particles in two dimensions with constant speed and a stochastic switching angle dynamics defined by a correlated dichotomous Markov process (telegraph noise) plus Gaussian white noise. We study various cases of the asymptotic diffusional motion of the particle which is characterized by the effective diffusion coefficient. Expressions for this coefficient are derived and discussed in dependence on the correlation time and the intensity of the noise. The situation with a given mean curvature is of special interest since a non-monotonic behavior of the effective diffusion coefficient as function of the noise intensity and correlation time is found. A timescale matching condition for maximal diffusion is formulated.

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Introduction. We study self-propelled particles moving in two dimensions at a constant speed v_0 . The time-dependent position $\vec{r}(t)$ of a particle follows from integrating its velocity $\vec{v}(t) = (v_x(t), v_y(t)) = v_0 (\cos \phi(t), \sin \phi(t))$ with time-dependent orientation $\phi(t)$. The orientation of the velocity vector at time t is governed by stochastic dynamics. We assume that $\phi(t)$ changes due to a constant torque superimposed by an unbiased dichotomous Markov process (DMP) $\zeta(t)$ which increases or decreases the local curvature of the particle's trajectory. In addition, a Gaussian white noise is present, corresponding to the thermal or environmental noise of the system.

Physically, this dynamics is motivated as an approximation to recently measured bimodal distributions $P(\Delta, \tau)$ of turning angles $\Delta = \phi(t + \tau) - \phi(t)$ during time τ , as observed in experiments with the zooplankton *Daphnia* [1–3], which is also able to sustain a constant mean speed over large time scales. By using a DMP, we approximate the bimodal structure by two delta peaks at $(\Omega + A_-)\tau$ and $(\Omega + A_+)\tau$, where A_- and A_+ denote the DMP strokes and Ω is the additional torque-induced angular velocity, which we will simply term “torque” in what follows. The constant torque can be motivated by various biological realizations. On the one hand, there are typical swarming characteristics which can be introduced by an effective torque [1, 4]. On the other hand, an asymmetric muscularity [5], an external magnetic field [6] or corresponding asymmetric boundary conditions [7] can lead to an effective torque as well.

Thus, our system is fully described by the constant speed $|\vec{v}| = v_0$, and the angle dynamics

$$\dot{\phi}(t) = \Omega + \zeta(t) + \xi(t). \quad (1)$$

$\xi(t)$ is the Gaussian white noise with zero mean and noise intensity D_ξ .

The dichotomous Markov process $\zeta(t)$ is time-homogeneous and switches between the two values A_+ and A_- with transition rates r_+ and r_- [8, 9]. r_+ denotes the rate for changing from A_+ to A_- and r_- denotes the rate of passage from A_- to A_+ . The mean value of this process, i.e. $\langle \zeta(t) \rangle = (r_- A_+ + r_+ A_-)/(r_+ + r_-)$, is fixed

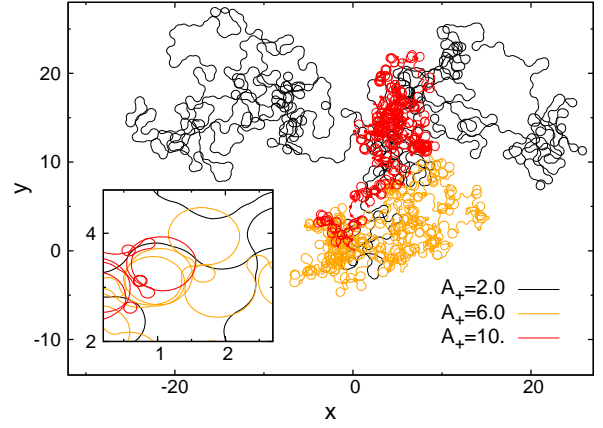


FIG. 1. (Color online) Spatial trajectories for dichotomous angular dynamics (see Eq.(1); details are shown in the inset) with $D_\xi = 0.01$, $v_0 = 1$, vanishing torque $\Omega = 0$, and the total duration $t_l = 10$. Two parameters of the DMP are fixed to be $A_- = -2$, $r_+ = 2$, A_+ varies, and r_- is defined by the vanishing mean of the DMP drive.

at zero in what follows, since the corresponding mean value can always be incorporated into the term Ω .

At first, we derive an explicit expression for the effective diffusion coefficient in our system which is defined as the long-time limit

$$D_{\text{eff}} = \lim_{t \rightarrow \infty} \frac{\langle (\vec{r}(t) - \vec{r}_0)^2 \rangle}{4t}. \quad (2)$$

We then discuss its dependence on the DMP parameters and on the external noise for the case without torque as well as for the case with non-vanishing torque Ω . We observe a torque-induced non-monotonic behavior of the effective diffusion coefficient, similar to the one recently discussed in the context of a system which is driven by an Ornstein-Uhlenbeck process (OUP) [10], as well as to the peaked diffusion of spiral waves driven by a correlated random forcing [11]. Finally, we study the DMP limits leading to shot noise and to Gaussian white noise [12]. The latter reproduces the well known result which was previously derived in [10, 13, 14].

Analytical considerations of a DMP-driven agent. In Fig.1, we present the spatial trajectories as obtained from simulations of our dynamics [Eq.(1)] for vanishing torque, small noise intensity D_ξ and different parameters of the dichotomous noise. The trajectories clearly show the dominant circular structures, as well as the strong influence of the parameters of the DMP on the particle's displacement. The effective diffusion coefficient of particles moving at a constant speed v_0 , can be written with the Taylor-Kubo relation as

$$D_{\text{eff}} = \frac{1}{2} \int_0^\infty \langle \vec{v}(t) \vec{v}(t+\tau) \rangle d\tau = \frac{v_0^2}{2} \int_0^\infty \langle \cos(\Delta(\tau)) \rangle d\tau \\ = \frac{v_0^2}{2} \text{Re} \left(\int_0^\infty \int_{-\infty}^\infty e^{i\Delta} P(\Delta, \tau) d\Delta d\tau \right), \quad (3)$$

where $P(\Delta, \tau)$ denotes the probability for an angle increment Δ during time τ . Using a continuous-time gen-

eralization of the classical persistent random walk, as it was studied in [15], we derive the probability to finish a dichotomous Markov step at a certain time with a certain angle. This calculation can be explicitly done [2] and leads to a coupled system of integral equations, which can be solved in an algebraic way by considering the Fourier-Laplace transform

$$\tilde{P}(k, s) = \int_{-\infty}^\infty e^{ik\Delta} \int_0^\infty e^{-s\tau} P(\Delta, \tau) d\tau d\Delta \quad (4)$$

of the corresponding probability density. With $\tilde{P}(k, s)$, Eq.(3) can be rewritten as

$$D_{\text{eff}} = \frac{v_0^2}{2} \text{Re} (\tilde{P}(k=1, s=0)), \quad (5)$$

and we are finally able to derive the effective diffusion coefficient of our dynamics

$$D_{\text{eff}} = \frac{v_0^2}{2} \left[\frac{(D_\xi + r_+ + r_-)(D_\xi(D_\xi + r_+ + r_-) - (A_+ + \Omega)(A_- + \Omega))}{(D_\xi(D_\xi + r_+ + r_-) - (A_+ + \Omega)(A_- + \Omega))^2 + ((A_+ + \Omega)(D_\xi + r_-) + (A_- + \Omega)(D_\xi + r_+))^2} \right. \\ \left. + \frac{\left(\frac{A_+ r_+ + A_- r_-}{r_+ + r_-} + \Omega \right) ((A_+ + \Omega)(D_\xi + r_-) + (A_- + \Omega)(D_\xi + r_+))}{(D_\xi(D_\xi + r_+ + r_-) - (A_+ + \Omega)(A_- + \Omega))^2 + ((A_+ + \Omega)(D_\xi + r_-) + (A_- + \Omega)(D_\xi + r_+))^2} \right]. \quad (6)$$

Without loss of generality we assume that $A_+ > 0 > A_-$ and $|A_+| > |A_-|$ which also implies $r_+ > r_-$. For describing the DMP dynamics in a more intuitive way, we use the following parameters [16]

$$A = A_+ - A_-, \quad \tau_c = \frac{1}{r_+ + r_-}, \quad 0 < p = -\frac{A_-}{A_+} \leq 1. \quad (7)$$

Now, A measures the strength of the process and τ_c is the correlation time of the DMP. The parameter p controls the asymmetry of the driving, the symmetric case corresponds to $p = 1$, and it tends to 0 for strongly asymmetric strokes.

Rewriting Eq.(6) in terms of these new parameters, we get

$$D_{\text{eff}} = \frac{v_0^2}{2} \cdot \frac{\left(D_\xi + \frac{1}{\tau_c} \right) \left(D_\xi^2 + \frac{D_\xi}{\tau_c} + \frac{pA^2}{(1+p)^2} - \frac{\Omega A(1-p)}{1+p} - \Omega^2 \right) + \left(\frac{1-p}{1+p} A + \Omega \right) \left(\frac{\Omega}{\tau_c} + 2\Omega D_\xi + \frac{D_\xi A(1-p)}{1+p} \right)}{\left(D_\xi^2 + \frac{D_\xi}{\tau_c} + \frac{pA^2}{(1+p)^2} - \frac{\Omega A(1-p)}{1+p} - \Omega^2 \right)^2 + \left(\frac{\Omega}{\tau_c} + 2\Omega D_\xi + \frac{D_\xi A(1-p)}{1+p} \right)^2}. \quad (8)$$

Discussion of the effective diffusion coefficient. Considering a vanishing torque $\Omega = 0$, we can discuss the symmetric and asymmetric limit of our DMP drive. A strong asymmetry leads with Eq.(8) to

$$\lim_{p \rightarrow 0} D_{\text{eff}} = \frac{v_0^2}{2D_\xi}, \quad (9)$$

so that we receive a divergent expression for a vanishing intensity of the thermal noise $\xi(t)$. Hence, the noise leads to the maintenance of the diffusive character of our system. This limit coincides with the result of the Gaussian white noise-driven angle discussed in [17]. In the

limit $p \rightarrow 0$, the DMP-term $\zeta(t)$ becomes negligible for the simple reason that the torque A_- vanishes. Therefore, only the Gaussian force $\xi(t)$ drives the angle which results in Eq.(9).

The symmetric, torqueless case $p = 1$ on the other hand, reproduces the result

$$\lim_{p \rightarrow 1} D_{\text{eff}} = \frac{v_0^2}{2} \cdot \frac{D_\xi + \frac{1}{\tau_c}}{D_\xi^2 + \frac{D_\xi}{\tau_c} + \frac{A^2}{4}}, \quad (10)$$

which has been previously derived in [2]. Analyzing this expression leads to a noise intensity which maximizes the effective diffusion coefficient, namely the value which

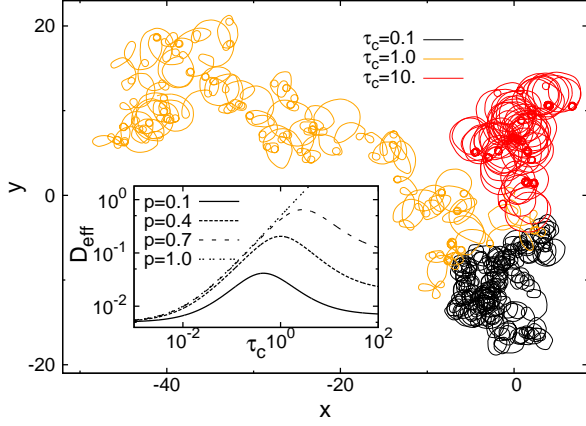


FIG. 2. (Color online) Spatial trajectories for different τ_c values and a non-vanishing torque $\Omega = 1$; the other parameters are $p = 0.4$, $D_\xi = 0.01$, $A = 2$, $v_0 = 1$, and total duration $t_l = 10$; the inset illustrates the effective diffusion coefficient D_{eff} versus correlation time τ_c for different asymmetry parameters p .

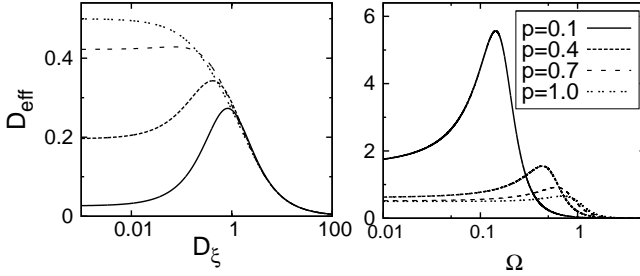


FIG. 3. Effective diffusion coefficient versus noise intensity D_ξ (left) and torque Ω (right) for different asymmetry parameters p within the theory (cf. Eq.(8)) for the constant torque $\Omega = 1$ (left), $D_\xi = 0.01$ (right), $\tau_c = 1$, $v_0 = 1$, $A = 2$.

obeys $D_\xi + 1/\tau_c = A/2$. The two dissipative time scales of the noises sources, i.e. $\tau_\xi = 1/D_\xi$ and τ_c , relate to the torque of the DMP which turns the particle.

A non-vanishing torque $\Omega \neq 0$ changes the basic characteristics of our dynamics and induces a systematic change of the curvature of the path. Figure 2 shows spatial trajectories for different values of τ_c and illustrates the influence of Ω . The fact that this influence on D_{eff} leads to a non-monotonous parameter dependence is demonstrated in Fig.2(inset) and 3. We recognize that D_{eff} as a function of τ_c , D_ξ and Ω possesses a well-defined maximum (a similar p dependence is not shown here).

This can be understood in view of the corresponding behavior of the agent considered [see Fig.2]. Small and large correlation times lead to a curled structure where the system stays in a certain DMP state either too long or too short in order to perform a considerable spatial displacement. In the case of $\tau_c = 10$, the particle performs a persistent circular motion with small curvature ($\Omega + A_-$) interrupted by small spins with large curva-

ture ($\Omega + A_+$) and the additive noise causes diffusion by shifting the centers of the circles stochastically. For the case with $\tau_c = 0.1$, fast DMP-switches induce an erratic motion. In contrast to the torqueless situation, the non-vanishing Ω reduces the displacement of the particle. On average, the motion follows again randomized circular lines determined by the non-vanishing torque and the fast DMP strokes, whose mean influence disappears for fast switchings. Calculating the limits of large and small τ_c values analytically, results in a non-zero value of D_{eff} in both cases due to the additive Gaussian noise. This property is seen in Fig.2 where both asymptotics tend to finite values.

The optimal τ_c in between the two limits, i.e. the state of maximal diffusion, corresponds to a maximally stretched trajectory for given values A_\pm and only the A_- stroke can decrease the curvature. Thereby, it can induce longer excursions which become maximal if the mean waiting time $1/r_-$ matches the time which the angle ϕ needs to rotate over half of a circle during the A_- stroke. Hence, the parameters have to obey $|A_- + \Omega|/r_- = \pi$ and for the notation introduced in Eq. (7) follows

$$\tau_c^{\text{max}} = \frac{\pi p}{|\Omega(1+p) - pA|}. \quad (11)$$

This result is in good agreement with the peaks in Fig.2 for small p values. It fails for larger p , where the influence of the A_+ stroke is not negligible.

The behavior of D_{eff} as a function of Ω [Fig.3(right)] shows a peak due to similar reasons. The case $\Omega = |A_-|$ causes straight paths within the corresponding DMP mode and will therefore enhance the spread. The peak in the dependence of D_{eff} on the noise intensity D_ξ in Fig.3(left) shifts for growing p to smaller noise values. Such a peak was already reported for similar dynamics but in the absence of a DMP [13, 14]. It is in agreement with our previous discussion of Eq.(9) that the effect of the DMP strokes disappear if $p \rightarrow 0$.

Taking the derivative of Eq.(8) with respect to τ_c leads, after some straightforward calculations, to a lengthy analytical result. τ_c values which maximize the diffusion coefficient are presented in Fig.4. It shows a perfect agreement with simulation results and with the peaks in Fig.2. The rough approximation given by Eq.(11) turns out to be rather good. The peaks in Fig.4 occur because of the mentioned rectilinear motion for $\Omega = |A_-| = pA/(1+p)$, since an increase of τ_c also increases the duration of the rectilinear motion and enlarges consequently D_{eff} . That is why the peak for $p = 1$ is shifted to infinitely large correlation times in Fig.2(inset).

The white shot noise limit of the DMP drive can be found by considering the limits $A_+ \rightarrow \infty$ and $r_+ \rightarrow \infty$ while the ratio $A_+/r_+ = -A_-/r_- = w$ holds constant [12]. The corresponding locomotion of the agent consists of a circular motion with mean curvature ($\Omega + A_-$) interrupted by infinitely fast turnings of the angle, induced by the short, large strokes A_+ . The autocorrelation function of $\zeta(t)$ in this shot noise limit reads

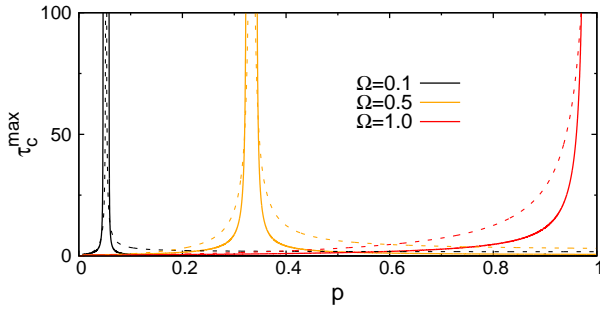


FIG. 4. (Color online) τ_c value which maximizes the effective diffusion coefficient versus asymmetry parameter p for different mean torques Ω ; the results of the full theory in Eq.(8) (solid lines) and for the approximative expression in Eq.(11) (dashed lines) are shown for the noise intensity $D_\xi = 0.01$ and the stroke strength $A = 2$.

$\langle \zeta(t)\zeta(t+\tau) \rangle = (-A_-w/\tau_c) \exp(-|\tau|/\tau_c)$ and therefore implies the noise intensity $D_\zeta = -A_-w$, while $A_- < 0$ holds. Rewriting of Eq.(6) within the mentioned limits leads to the effective diffusion coefficient

$$D_{\text{eff}}^{\text{shot}} = \frac{v_0^2}{2} \cdot \frac{D_\xi + D_\zeta + D_\xi w^2}{(D_\xi + D_\zeta - \Omega w)^2 + (D_\xi w + \Omega)^2}. \quad (12)$$

Here, $D_{\text{eff}}^{\text{shot}}$ becomes maximal for $\Omega = -A_-w^2/(1+w^2)$, which implies $\Omega \leq |A_-|$.

The white Gaussian limit of the DMP drive can be derived by considering the limit $w \rightarrow 0$ while D_ζ is hold constant [12]. Doing so in Eq.(12), we find

$$D_{\text{eff}}^{\text{Gauss}} = \frac{v_0^2}{2} \cdot \frac{D_\xi + D_\zeta}{(D_\xi + D_\zeta)^2 + \Omega^2}. \quad (13)$$

If we introduce a total noise intensity $D = D_\xi + D_\zeta$, this expression coincides with the result in [10, 13, 14] for an agent under influence of Gaussian white noise with intensity D . Maximal diffusion at the value $D_{\text{eff}}^{\text{Gauss}} = v_0^2/(4\Omega)$ is obtained for the total noise intensity $D = \Omega$. Thus, the resonance occurs where D , i.e. the angular correlation decay rate in the case of a Gaussian white angle drive, equals the effective torque, as it is likewise the case in Eq.(10) with the additional decay rate of the correlation within the DMP drive.

Conclusion. We have discussed exact results for the effective diffusion coefficient of a particle moving at a constant speed under influence of a constant torque, dichotomous angular Markov noise and additional directional Gaussian perturbations. The results help to understand the behavior of our system in a qualitative and quantitative way. They clarify the role of the asymmetry and of an additional torque in the DMP-driven angle dynamics. The strongly peaked bimodal angular probability distribution, which we have assumed in our model, is of course an enormous simplification of the ones found in real biological systems. But in view of the bulk of works discussing symmetric angular distributions [2, 17–20], it seems to be reasonable to discuss the influence of a certain asymmetry as well. Since the diffusion coefficient is one of the most easily accessible quantities in experiments, we hope that this work not only fills a gap in our general theoretical understanding of self-propelled agents, but will also stimulate corresponding experimental studies.

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